OBJECT AND IDEAL
A Mathematical History of Hyperbolic Space

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We have built a world of rectilinearity—the rooms we inhabit, the skyscrapers we work in, the gridlike arrangements of our streets, the freeways we cruise on our daily commute speak to us in straight lines. We have learned to play by Euclidean rules because two thousand years of geometric training have engraved the grid in our minds. But in the early nineteenth century mathematicians became aware of a space in which lines cavorted in aberrant formations, suggesting the existence of a new geometry.

To all at the time *hyperbolic space* seemed pathological, for it contradicted the axioms of Euclid, overthrowing millennia of mathematical wisdom and offending common sense. "For God's sake, please give it up. Fear it no less than the sensual passions, because it, too, may take up all your time and deprive you of your health, peace of mind and happiness in life," wrote Farkas (Wolfgang) Bolyai to his son János; both were mathematicians at the forefront of investigating this bizarre spatial construct. Carl Friedrich Gauss, the prince of mathematicians, kept his studies private: "I fear the howl of the Boetians, if I make my ideas known," he confided to a friend.

The apprehension instilled in mathematicians by the revelation of hyperbolic space heralded the beginning of a new era in the discipline's history and a commensurate revolution in thinking about what mathematics means and its relationship to the world of objects. The distant howlings Gauss intuited were the cries of pain that would soon erupt as mathematicians tore their subject from its anchoring in "real" physical things and set it free as an epistemic bubble untethered from materialist contingencies.

The Greeks, of course, had laid down the original schema: Pythagoras, Plato, and Aristotle all agreed that mathematics was the study of *quantity* and *form*. Quantity is made known to us through the act of counting—one, two, three, four. Its elements, the numbers, were (to the Greeks) the Platonic ideals that stood behind the material manifestations of, say, four oranges or four chairs. While things decay, the number itself was that which endured, the incorruptible, immutable core that in Pythagorean philosophy transcended its every embodiment: four flowers will wilt, four men will die, four mountains will eventually be worn away, but Four, the ideal, is forever.

Form is what we study through the discipline of geometry; its elements are points and lines, triangles, squares, circles, and so on. Again the Greeks believed that this realm of ideals was mediated by concrete things: the flat plane of a tabletop approximated the Euclidean plane, the surface of the earth approximated a sphere, a circle could be approximated with a compass. To the Greek way of thinking, mathematical ideals such as numbers and circles existed in a transcendent realm above and beyond the material plane. Such ideal constructs served as the models for the imperfect realm of objects, which strive as best they can against the contingencies of substance to realize the perfections with which they are imprinted. Thus while ideals literally in-form objects, one of the functions of objects in the Pythagorean/Platonic tradition was to guide our thoughts upward to the realm of ideals.

Modern physicists continue this philosophical thread when they propose that mathematics is the language in which nature's laws are written and that the study of such laws, along with the things that embody them, will lead us, in Stephen Hawking's famous phrase, toward "the mind of God."

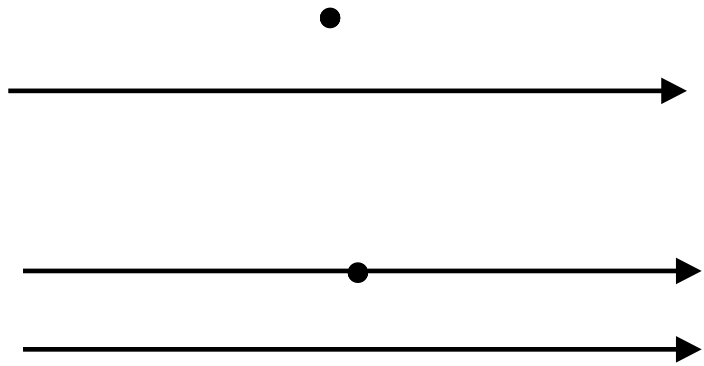
But what happens if the correspondence between things and ideals is broken? What if ideals have a life of their own?

The study of number hinted first at this unsettling possibility. What are we to make, for instance, of the perfectly legitimate mathematical operation "one minus one"? For hundreds of years European mathematicians resisted the notion of "zero," an idea they had encounterd in Indian mathematics. How can we signify nothing, which, by definition, doesn't exist? Negative numbers compounded the problem, for one cannot have -4 oranges. A preoccupation with thingness militates against extending the number system, yet someone in debt may well be worth -4 dollars. Ultimately it was the spread of double-entry bookkeeping in the twelfth and thirteenth centuries that cemented zero and the negatives into the Western mathematical scheme.

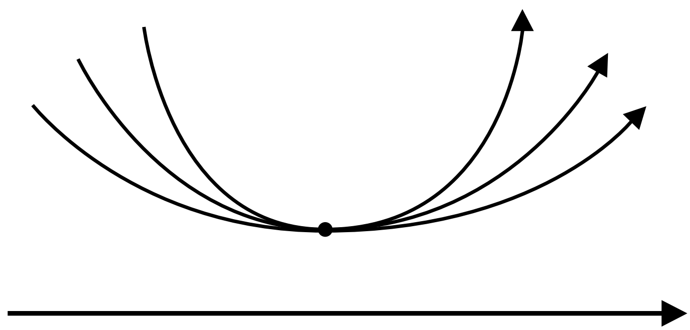
With the door thus opened, numbers soon revealed properties undreamed of by the bean counters. *Imaginary* numbers (the square roots of the negatives) and *complex* numbers (compounds of the reals and imaginaries) foisted themselves on the bewildered consciousness of late Renaissance minds. What possible meaning could be attached to "the square root of minus one"? The very name "imaginary" testified to the bamboozlement mathematicians felt; it was as if they were dealing with dream creatures, the formal equivalent of unicorns.

Hyperbolic geometry was the first glimpse that similar conundrums would occur in the realm of form. Here we begin with parallel lines: what does it mean to say that two lines are parallel? In Euclidean geometry the term connotes two lines that never meet. Take a line and a point outside this line: how many other lines can you draw through the point that never meet the original line (FIG. 22)? The answer is one. This indeed may serve as a definition of the Euclidean plane, for here the proposition holds for *all* lines. But another geometric possibility inheres in the form of the sphere. Consider the surface of the earth; all lines of longitude are parallel at the equator, yet all meet at the North and South poles. In spherical geometry there is no such thing as straight lines that never meet. Here all parallels converge.

Parallelism is thus a more complex concept than might first be supposed, and in the nineteenth century mathematicians



[Fig. 22]   Parallel lines in Euclidean space



[Fig. 23]   Parallel lines in hyperbolic space realized that it encompassed a third, bizarre possibility. From a purely formal perspective it is quite legitimate to propose a geometric space in which an infinite number of parallel lines may pass through a single point yet never intersect with an original line (FIG. 23). In homage to this excess, mathematicians named the resulting construct "hyperbolic space." Here the ideal concept of parallel lines became uncoupled from any apparent referent in the material realm. Mathematicians were bamboozled, astounded, and quite literally appalled.

We know Euclidean space, or we think we do, for we are constantly making grids and graphs and rectilinear rooms, and we know the surface of a sphere, for we live on one. But how can we make sense of hyperbolic geometry? One way of understanding this is through the concept of convergence: where a sphere represents a geometry in which parallel lines converge, on a hyperbolic plane they *diverge*. Another approach, as Gauss perceived, is to think in terms of curvature: a sphere is a space with positive curvature, and a Euclidean plane has zero curvature; the hyperbolic plane is simply a space with negative curvature. It is the geometric equivalent of a negative number. *Plus*, *minus*, and *zero* may thus be rendered into geometric terms, each identifying a different set of spatial relationships. In a very powerful sense form and quantity themselves converge. Yet the price of this almost-mystical abstraction is a divorcing of mathematics from the "sensible" world of objects

Objects had led to ideals, but now ideals had taken over, for no one in the nineteenth century imagined that hyperbolic geometry might be realized in actual physical things. Practically speaking, that seemed absurd; the Boetians were howling at the gates. The one potential exception that impinged itself on nineteenth-century consciousness was the structure of cosmological space, and Gauss speculated that our universe may have hyperbolic form. The geometry of space on the universal scale still remains an open question in cosmology and one that space-based telescopes such as the Hubble are currently striving to answer. Most evidence points to a

Euclidean universe, yet there is intriguing data to suggest that we may live in a hyperbolic world.

The relationship between mathematical ideals and material facts has itself been an open question since at least the seventeeth century, when René Descartes presciently asked, Is all of mathematics realized in the realm of objects? Descartes's answer was yes. Most mathematicians today might hesitate to respond so affirmatively, however, for in the latter half of the nineteenth century their understanding of the discipline underwent a radical transformation due to discoveries about such concepts as hyperbolic space and imaginary numbers.

By the 1860s a new philosophy had begun to emerge that would liberate mathematics entirely from its material moorings. According to Augustus De Morgan, mathematical concepts need not refer to anything physically existing. In De Morgan's terms, mathematics was purely a "science of symbols" whose only requirement was that its logic be self-consistent. In short, mathematics did not intrinsically reference anything. Its referential quality, when that existed, was outside the domain of mathematics per se and of no concern to practicing mathematicians.

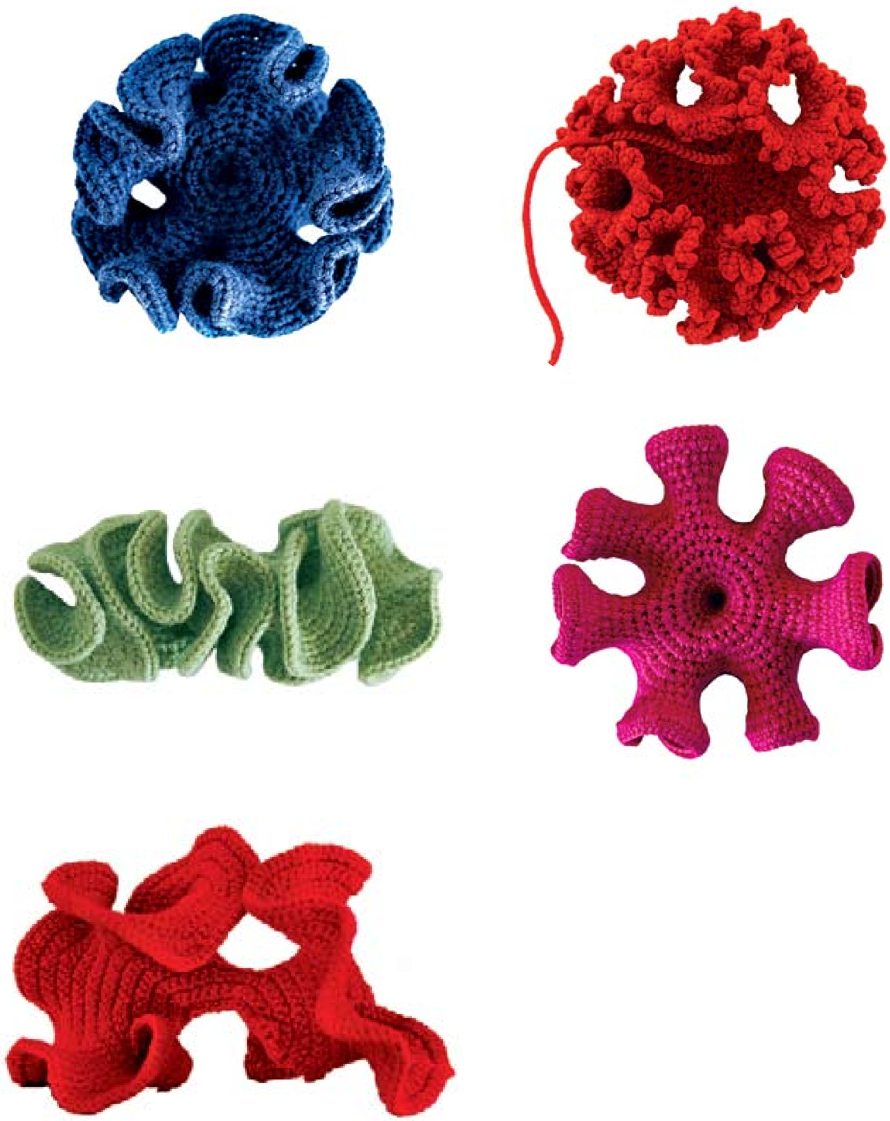
To mathematicians of the past 150 years, 4 and -4 and 4i are equally valid numbers; it matters not if they have material instantiations. Much of the project of mathematics over the past century and a half has been a steady process of abstraction as more and more branches have been transformed into purely symbolic terms. Thus the study of numbers gave rise to the discipline of modern algebra, with its dazzling taxonomy of *groups* and *rings* and *fields*, and the thousands of species of these genera that constitute a sort of formalist tree of life. Meanwhile the study of form begat non-Euclidean geometry, finite geometry, and Riemannian geometry, plus the field of topology (where a coffee cup and a doughnut are one and the same).

But in what we might call a return of the repressed, the object-sphere uncannily has bubbled back as many of the most seemingly abstruse mathematical concepts have turned out to have real material analogs. Imaginary numbers, for instance, play a central role in the design of electrical circuits: cell phones, radio receivers, and WiFi stations must each be tuned to electromagnetic frequencies via circuit elements that tweak the imaginary part of the current and voltage. Mathematicians may have been horrified by hyperbolic geometry, but nature grasped its potential in the Ordovician age: corals, kelps, sponges, nudibranchs, and sea slugs all exhibit hyperbolic anatomical features. For filter-feeding organisms, hyperbolic surfaces offer an ideal solution to the problem of lunch by maximizing surface area in a given volume.

Though mathematicians had long believed that it wasn't possible to make physical models of hyperbolic space, in 1997 a Latvian professor named Daina Taimina realized how to construct this geometric ideal using the craft of crochet. At the Institute For Figuring, we have explored this technique over the past four years, elaborating on Dr. Taimina's methods to articulate a new ecology of forms (Fig. 24).

Crocheted models of hyperbolic space, hooked together from animal hairs and vegetable fibers, are a material embodiment of a symbolic ideal long thought to be logically untenable. Soft and pliable, fluffy and hairy, made by female hands, these models call to mind the Red Queen's advice to Alice that with enough practice she too could develop the skill of believing "six impossible things before breakfast." Alice's creator, Charles Dodgson, aka Lewis Carroll, was himself a younger contemporary of De Morgan's who also worked in the emerging field of mathematical logic. Through Carroll's pen the howl of the Boetians was transformed into a trip through Wonderland, the theoreticians' cries of pain commuted into a smirk that lingered in the air like the grin of a Cheshire Cat.

The models crocheted using these techniques not only materialize but also temporalize hyperbolic space, for these works are brought into being through the iterative act of repeating a simple sequence of steps again and again and again. Each hyperbolic model advances by increasing crochet stitches according to a primitive



[Fig. 24] The Institute For Figuring; various models of hyperbolic space, 2007–2008; cotton, polyester yarn; dimensions variable algorithm: make "n" stitches, then increase by one; repeat ad infinitum (or until you've had enough).

Generated through time-based labor, these forms are almost impossible to model on computers or to re-create through mechanical processes: while knitting has long been automated, there is no machine that can crochet. Moreover, hyperbolic structures cannot be modeled mathematically by analytic equations—in short, they are not subject to the interpretive techniques of calculus. The only way to know the final form of an iterative structure is to literally *play* it out. Such process-driven actualization also underlies images of fractals, which themselves constitute a further revolution in geometry: these are structures that possess a fractional dimension, another seemingly paradoxical concept. In order to know the Mandelbrot Set and other fractal structures, we have no choice but to engage with them temporally, to let a computer iterate their underlying primitives over and over again. Here too the element of time overcomes an idealized limitation (the concept of a "dimension"), bringing into being configurations previously deemed absurd or even impossible.

With fractals the temporal factor is played out by computers; with crochet models it must be provided by personal human commitment. This effort—intimate in scale, domestic in setting, usually female in practice—literally instantiates a hitherto purely imagined form. The body becomes the conduit for the realization of the mind's most abstract flight; matter becomes the medium for the intellect's mathematical message. Bridging the division between time and space, theory and practice, matter and process, a woman's hands convert the thread of a lamb into a spatial conundrum. Here object and ideal converge.